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Let $b: [-1,0] \to \mathbb{R}$ be a C^2-function, $b(\theta) > 0$, $\theta \in (-1,0]$, $b(1) = 0$, $b'(\theta) \geq 0$, $b''(\theta) \geq 0$, $\theta \in [-1,0]$ and there is a θ_0 such that $b''(\theta_0) > 0$. Suppose $g: \mathbb{R} \to \mathbb{R}$ is a C^1-function such that $\int_0^x g(s)ds \to \infty$ as $|x| \to \infty$ and consider the equation

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ITEM #20, CONT.:

$$\dot{x}(t) = -\int_{-1}^{0} b(\theta)g(x(t+\theta))d\theta$$

Every solution of this equation approaches a zero of g. If the zeros of g are bounded, there is a maximal compact invariant set $A_{b,g}$ of this equation in $C([-1,0], \mathbb{R})$ which is one dimensional and consists only of the zeros of g and the unstable manifolds of these zeros. If g has only one zero, then $A_{b,g}$ is a point. If g has no more than three simple zeros, then the set $A_{b,g}$ is simply an arc with the unstable zero connected to the stable ones. In the class of g which have five simple zeros, we show that there are five distinct ways that the zeros of g can be connected by orbits in $A_{b,g}$. Only one of these preserves the order of the zeros on the reals. This shows clearly the importance of considering the set $A_{b,g}$ and the structure of the flow on this set rather than just asserting that every solution approaches a zero of g.

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On a gradient-like integro-differential equation

by

Jack K. Hale and Krzysztof P. Rybakowski

Abstract

Let $b: [-1,0] \to \mathbb{R}$ be a C^2-function, $b(\theta) > 0$, $\theta \in (-1,0]$, $b(1) = 0$, $b'(\theta) \geq 0$, $b''(\theta) \geq 0$, $\theta \in [-1,0]$ and there is a θ_0 such that $b''(\theta_0) > 0$. Suppose $g: \mathbb{R} \to \mathbb{R}$ is a C^1-function such that $\int_0^x g(s)ds \to \infty$ as $|x| \to \infty$ and consider the equation

$$\dot{x}(t) = -\int_{-1}^{0} b(\theta)g(x(t+\theta))d\theta$$

Every solution of this equation approaches a zero of g. If the zeros of g are bounded, there is a maximal compact invariant set $A_{b,g}$ of this equation in $C([-1,0], \mathbb{R})$ which is one dimensional and consists only of the zeros of g and the unstable manifolds of these zeros. If g has only one zero, then $A_{b,g}$ is a point. If g has no more than three simple zeros, then the set $A_{b,g}$ is simply an arc with the unstable zero connected to the stable ones. In the class of g which have five simple zeros, we show that there are five distinct ways that the zeros of g can be connected by orbits in $A_{b,g}$. Only one of these preserves the order of the zeros on the reals. This shows clearly the importance of considering the set $A_{b,g}$ and the structure of the flow on this set rather than just asserting that every solution approaches a zero of g.

1. Introduction

Let $b: [-1,0] \to \mathbb{R}$ be a C^2-function such that $b(\theta) > 0$, $\theta \in (-1,0]$, $b(-1) = 0$, $b'(\theta) \geq 0$, $b''(\theta) \geq 0$ for $\theta \in [-1,0]$ and there exists a $\theta_0 \in [-1,0]$ such that $b''(\theta_0) > 0$. Suppose $g: \mathbb{R} \to \mathbb{R}$ is a C^1-function and consider the equation

$$\text{(1.1)(b,g)} \qquad \dot{x}(t) = -\int_{-1}^{0} b(\theta)g(x(t+\theta))d\theta$$

For any $\phi \in C = C([-1,0], \mathbb{R})$, there is a unique solution $x(\phi)$ through ϕ at $t = 0$. If $T_f(t): C \to C$ is defined by $[T_f(t)\phi](\theta) = x(\phi)(t+\theta)$, then $T_f(t)$, $t \geq 0$, is a strongly continuous semigroup on C. We think of the solution $x(\phi)$ of (1.1)(b,g) as defining a curve $\{T_f(t)\phi, t \geq 0\}$ in C and, therefore, can consider the geometric concepts of ω-limit sets, α-limit set and invariant set (see [1]).

Suppose $\int_0^x g(s)ds \to \infty$ as $|x| \to \infty$. It is known (see [1,2]) that the ω-limit set of every solution of (1.1)(b,g) is a zero of g and, also, the α-limit set of any nonconstant bounded solution of (1.1)(b,g) is an unstable zero of g. If a is a zero of g then a is hyperbolic if and only if $g'(a) \neq 0$, uniformly asymptotically stable if $g'(a) > 0$ and unstable if $g'(a) < 0$. Furthermore, the unstable manifold $W^u(a)$ of a is one dimensional if a is unstable.

If the set of zeros of g is bounded, then there is a bounded set B such that every solution eventually enters B, that is, (1.1)(b,g) is point dissipative. It follows (see [1,2]) that there is a maximal compact invariant set $A_{b,g}$ for (1.1)(b,g) which is uniformly asymptotically stable and attracts bounded sets of C.

From the fact that the α-limit set of any nonconstant bounded solution is an unstable zero of g, it follows that $A_{b,g} = \bigcup\{W^u(a): g(a) = 0\}$ and $A_{b,g}$ is one dimensional. The purpose of this paper is to discuss in some detail the structure of the set $A_{b,g}$ for a fixed b and a certain class of g.

Let G_k be the class of all C^1-functions g satisfying the following conditions:

1) $\int_0^x g(s)ds \to \infty$ as $|x| \to \infty$.

2) g has exactly $2k+1$ zeros $a_1 < a_2 < \cdots < a_{2k+1}$ all of which are simple.

Let the topology on G_k be that generated by the seminorms $||g||_M = \sup_{x \in M}(|g(x)| + |g'(x)|)$, where M is a compact set in $\mathbb{R}$. For any $g \in G_k$, all zeros of g are hyperbolic and the zeros a_{2j}, $j = 1,2,\ldots,k$, are saddle points with unstable manifolds $W^u(a_{2j})$ one dimensional. Thus, for each a_{2j}, there are exactly two distinct orbits defined for $t \in (-\infty,\infty)$ whose α-limit sets are a_{2j}. We call these orbits emanating from a_{2j}. Fix b as above. Let $g,\tilde{g} \in G_k$ have $a_1 < \cdots < a_{2k+1}$ and $\tilde{a}_1 < \cdots < \tilde{a}_{2k+1}$, resp., as their zeros. Call g and $\tilde{g}$ equivalent $(g \sim \tilde{g})$ if for all $i,j \in \{1,\ldots,2k+1\}$, there is an orbit $x(t)$ of (1.1)(b,g) emanating from a_i and tending to a_j as $t \to \infty$ if and only if there is an orbit $\tilde{x}(t)$ of (1.1)(b,$\tilde{g}$) emanating from $\tilde{a}_i$ and tending to $\tilde{a}_j$ as $t \to \infty$. This clearly defines an equivalence relation on G_k. We say $g \in G_k$ is <u>~-stable</u> if the equivalence class of g is a neighborhood of g in G_k.

It is not difficult to show that g is ~ stable if the ω-limit set of every orbit in $A_{b,g}$ which is not a point is a stable zero of g;

that is, a point a_n, n odd, $1 \le n \le 2k+1$. Since $A_{b,g}$ is a global attractor and uniformly asymptotically stable, this is equivalent to saying that g is $\sim$-stable if the ω-limit set of every orbit of (1.1)(b,g) defined and bounded on $(-\infty,\infty)$ is a stable zero of g. If it were known that the map $T_{b,g}(t)$ is one-to-one on $A_{b,g}$, this latter statement would be equivalent to the following: there is a neighborhood V of g such that, for any $\tilde{g} \in V$, there is a homeomorphism of $A_{b,g}$ onto $A_{b,\tilde{g}}$ which preserves orbits and sense of direction in time; that is, g is <u>structurally stable</u> when the map $T_{b,g}(t)$ is restricted to $A_{b,g}$. We have not been able to prove that $T_{b,g}(t)$ is one-to-one on $A_{b,g}$ and this is the reason for taking the weaker definition of equivalence. If g is analytic, then $T_{b,g}(t)$ is one-to-one (see [1]).

The ultimate objective would be to describe the equivalence classes in G_k. The cases $k = 0,1$ are trivial. Suppose $k = 2$; that is, each $g \in G_2$ has five zeros $a_1 < a_2 < a_3 < a_4 < a_5$ with a_2, a_4 being saddle points, and a_1, a_3, a_5 being uniformly asymptotically stable. If a_j is an unstable equilibrium point with a_k, a_ℓ being the corresponding ω-limit sets of the orbits emanating from a_j, we designate this by $j[k,\ell]$. The structure of the flow on $A_{b,g}$ and the equivalence classes in G_2 are then determined by a pair $\{2[i,j],4[k,\ell]\}$ expressing the fact that the unstable manifold through a_2 has ω-limit set $\{a_i, a_j\}$ and the one through a_4 has ω-limit set $\{a_k, a_\ell\}$.

Our main result states there are exactly five equivalence classes in G_2; namely $\{2[1,3],4[3,5]\}$, $\{2[1,4],4[3,5]\}$, $\{2[1,5],4[3,5]\}$, $\{2[1,3],4[2,5]\}$, $\{2[1,3],4[1,5]\}$. The only class that preserves the natural order of the reals on $A_{b,g}$ is $\{2[1,3],4[3,5]\}$. The first, third and fifth cases are $\sim$-stable. The second and fourth cases have a connection between the saddle points a_2 and a_4. It seems plausible that these cases are not $\sim$-stable, but no proof is available.

The fact that five equivalence classes can occur indicates clearly the importance of studying the structure of the flow on $A_{b,g}$ rather than merely asserting that every solution of (1.1)(b,g) approaches a zero of g.

We have not characterized the equivalence classes in G_k for $k \geq 3$, but it should be possible to adapt the methods below to this case.

2. This section is devoted to the statement and proof of several lemmas.

Lemma 2.1. For every $\rho > 0$, $K > 0$, there is a $C_0 = C_0(K,\rho) > 0$ such that, for all C, $0 < C \leq C_0$, and all a, Δ, $|a| \leq K$, $|a| - 2|\Delta| \geq \rho$, the solution $y(n)$ of the difference equation

$$(2.1)\qquad \begin{aligned} y(n) &= y(n-1) - Cy(n-2), \quad n \geq 1 \\ y(0) &= a + \Delta, \; y(-1) = a \end{aligned}$$

satisfies $\operatorname{sgn} y(n) = \operatorname{sgn} a$ for $n \geq -1$.

Proof: Let $\lambda_i = \lambda_i(C)$ be the roots of the characteristic equation $\lambda^2 - \lambda + C = 0$. If $0 \leq C < 1/4$, then λ_1, λ_2 are real, nonnegative and distinct, say $\lambda_1(C) < \lambda_2(C)$. Moreover, $\lambda_i(C)$ is continuous in C with $\lambda_1(0) = 0$, $\lambda_2(0) = 1$. For the initial data specified in (1.2) and $n \geq -1$,

$$(\lambda_2-\lambda_1)y(n) = (a(1-\lambda_1) + \Delta)\lambda_2^{n+1} + (a(\lambda_2-1) - \Delta)\lambda_1^{n+1}$$

The remainder of the argument follows easily from the fact that $\lambda_1(0) = 0$, $\lambda_2(0) = 1$ and the hypotheses on a, Δ.

Lemma 2.2. For every $\rho > 0$, $K > 0$, there is an $m_0 = m_0(\rho,K)$ such that, for all $0 < m \leq m_0$ and $f(x) = mx$, the following properties hold:

1) If $\phi(\theta)$ is continuous, positive and nonincreasing on $[-1,0]$, $\phi(-1) \leq K$ and $\phi(-1) - 2[\phi(-1) - \phi(0)] \geq \rho$, then the solution $x(t)$ of (1.1)(b,f) through ϕ satisfies $0 < x(t) < \phi(0)$ for $t \in (0,\infty)$ and $x(t) \to 0$ as $t \to \infty$ monotonically.

2) If $\phi(\theta)$ is continuous, negative and nondecreasing on $[-1,0]$, $-\phi(-1) \leq K$ and $-\phi(-1) - 2(\phi(0) - \phi(-1)) \geq \rho$, then the solution $x(t)$ of (1.1)(b,f) through ϕ satisfies $\phi(0) < x(t) < 0$ for $t \in (0,\infty)$ and $x(t) \to 0$ as $t \to \infty$ monotonically.

Proof: Fix $\rho > 0$, $K > 0$ arbitrarily. With $C_0 = C_0(\rho,K)$ as in Lemma 2.1, define $m_0 = C_0/M$, $M = \int_{-1}^{0} b(\theta)d\theta$. Let $0 < m \leq m_0$ be arbitrary and define $C = mM$. Suppose ϕ satisfies the hypotheses in 1) and let $y(n)$ be the solution of (1.2) with $a = \phi(-1)$, $a + \Delta = \phi(0)$. From Lemma 2.1, $y(n)$ is decreasing, $0 < y(n) < \phi(0)$, $n \geq 1$, $y(n) \to 0$ as $n \to \infty$. Let $t_n \geq 0$ be the unique minimal $t_n \geq 0$ such that $x(t_n) = y(n)$ and set $t_{-1} = -1$, $t_0 = 0$. To prove the claim $0 < x(t) < \phi(0)$ in 1), it is sufficient to show that $t_{n-1} - t_{n-2} \geq 1$ for every $n \geq 1$. This inequality is true for $n = 1$. Assuming that it holds for some n, we obtain

$$
\begin{aligned}
-Cy(n-2) &= y(n) - y(n-1) = x(t_n) - x(t_{n-1}) \\
&= -\int_{t_{n-1}}^{t_n} \left(\int_{-1}^{0} b(\theta)mx(s+\theta)d\theta\right)ds \\
&\geq -(t_n-t_{n-1})Cx(t_{n-1}-1) \\
&\geq -(t_n-t_{n-1})Cx(t_{n-2}) = -(t_n-t_{n-1})Cy(n-2)
\end{aligned}
$$

This implies $t_n - t_{n-1} \geq 1$ and proves $0 < x(t) < \phi(0)$, $t > 0$. Thus $x(t)$ is decreasing for $t > 0$ and approaches zero as $t \to \infty$. Case 2) is proved by replacing x by $-x$.

Lemma 2.3. Let α_1,α_2,m_2 and K be real numbers, $\alpha_1 \neq \alpha_2$, $m_2 < 0$, $K > 0$. Let $I = [\alpha_1,\alpha_2]$ if $\alpha_1 < \alpha_2$ and $I = [\alpha_2,\alpha_1]$ if $\alpha_2 < \alpha_1$. Then there is a mapping $f\colon I \to \mathbb{R}$, $f \in C^1(I)$, $|f| \leq K$ on I, $f'(\alpha_2) = m_2$, $f^{-1}\{0\} = \{\alpha_1,\alpha_2\}$, f is affine in a neighborhood of α_1 and α_2 and there is a function $x\colon \mathbb{R} \to \operatorname{Int} I$, $x \in C^1(\mathbb{R})$, such that $\lim_{t\to\infty} x(t) = \alpha_1$, $\lim_{t\to-\infty} x(t) = \alpha_2$, and x satisfies (1.1)(b,f).

Proof: Assume that the lemma is true for $\alpha_1 < \alpha_2$ and suppose $\alpha_2 < \alpha_1$. Let $\bar{\alpha}_1 = \alpha_2$, $\bar{\alpha}_2 = \alpha_1$. Then $\bar{\alpha}_1 < \bar{\alpha}_2$. Choose $\bar{f} = \bar{f}(\bar{\alpha}_1,\bar{\alpha}_2,m_2)$, $\bar{x} = \bar{x}(\bar{\alpha}_1,\bar{\alpha}_2,m_2)$ as in the statement of the Lemma. Define $f: [\alpha_2,\alpha_1] \to \mathbb{R}$, $f(x) = -\bar{f}(\alpha_1+\alpha_2-x)$ and $x(t) = \alpha_1 + \alpha_2 - \bar{x}(t)$. Then it is obvious that f and x satisfy the Lemma. Hence it is enough to prove the Lemma for $\alpha_1 < \alpha_2$.

Let $f_2(x) = m_2(x-\alpha_2)$. Then by the assumptions on b, there is a unique $\lambda_2 > 0$ such that $x(t) = \alpha_2 - e^{\lambda_2 t}$ is a solution of (1.1)(b,f_2) on $(-\infty,\infty)$. Let $K = 2(\alpha_2-\alpha_1)$, $\rho = (1/3)(\alpha_2-\alpha_1)$. Choose $m_0 = m_0(\rho,K)$ as in Lemma 2.2. For $0 < m \leq m_0$, $f_1(x) = f_1(m)(x) = m(x-\alpha_1)$ let

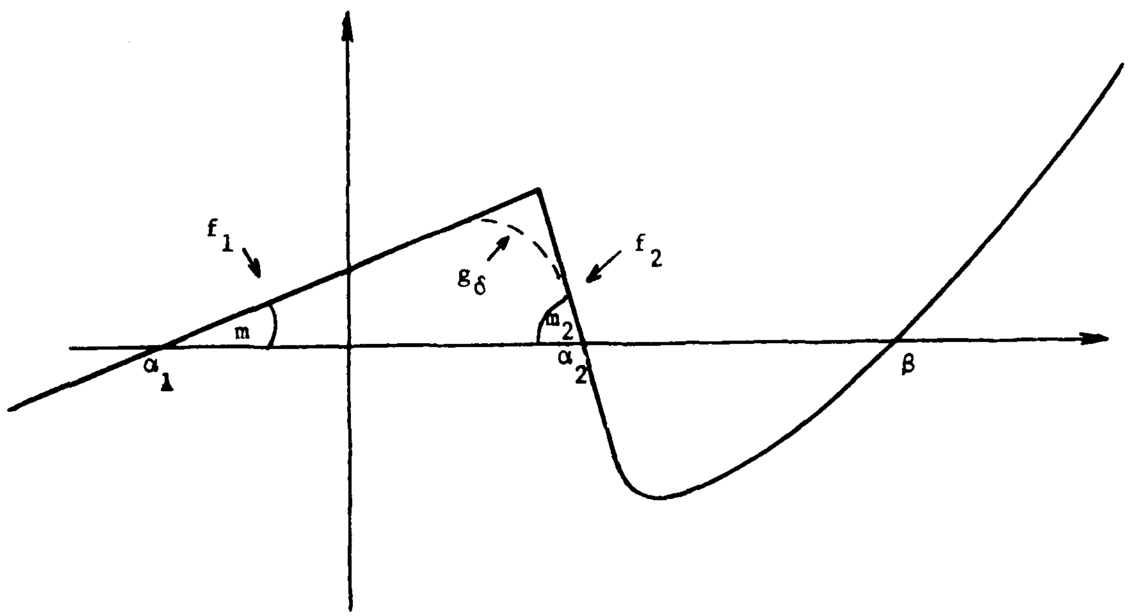

Figure 1

$\hat{x} = \hat{x}(m)$ be the unique coincidence point of f_1 and f_2 (cf. Fig. 1). Let $\hat{t} = \hat{t}(m)$ be the unique point such that $\alpha_2 - e^{\lambda_2 \hat{t}} = \hat{x}$. Define $h = h_m: \mathbb{R} \to \mathbb{R}$ so that $h = f_1$ on $(-\infty,\hat{x}]$, $h = f_2$ on $[\hat{x},\alpha_2]$ and h is arbitrary on (α_2,∞) but such that $h \in G_1$ and the zeros of h are: $\alpha_1 < \alpha_2 < \beta$ where β is some real number. Fix $\varepsilon > 0$ and α such that $\alpha_1 < \alpha < \alpha_2 - \varepsilon$, and $(\alpha_2-\varepsilon) - \alpha_1 \geq 2\rho + 2(\alpha_2-\alpha)$.

If $y(t)$, $t \geq 0$, is the solution of (1.1)(b,h) through $\phi(\theta) = \alpha_2 - e^{\lambda_2(\hat{t}+\theta)}$, define $z(t) = \alpha_2 - e^{\lambda_2 t}$ for $t < \hat{t}$, $z(t) = y(t-\hat{t})$ for $t \geq \hat{t}$. It follows that $z(t)$ solves (1.1)(b,h) on $(-\infty,\infty)$. Also, it is clear that there is a unique minimal $t' = t'(m)$ such that $z(t') = \alpha$. For m small enough $\hat{x} > \alpha_2 - \varepsilon$ and hence $t' > \hat{t}$. Moreover,

$$z(t') - \hat{x} = z(t') - z(\hat{t}) = -\int_{\hat{t}}^{t'} \left(\int_{-1}^{0} b(\theta)h(z(s+\theta))d\theta ds.\right.$$

Since $\hat{x} \geq z(t) \geq \alpha$ for $t \in [\hat{t},t']$, we have $0 \leq g(z(t)) = m(z(t)-\alpha_1)$ for such t, and this implies

$$|z(t')-\hat{x}| \leq M\cdot m(t'-\hat{t})(\hat{x}-\alpha_1),$$

where $M = \int_{-1}^{0} b(\theta)d\theta$. Hence,

$$0 < \alpha_2 - \varepsilon - \alpha < (\hat{x} - z(t')) \leq M\cdot m(t'-\hat{t})(x-\alpha_1),$$

i.e., $t'-\hat{t} \to \infty$ as $m \to 0$. Hence, for all m small enough, and all $t \in [\hat{t},\hat{t}+1]$, $\alpha < z(t) < \alpha_2$. Moreover, taking m smaller, if necessary,

$$\begin{aligned} &z(\hat{t}-1) - \alpha_1 - 2(z(\hat{t}-1) - z(\hat{t})) \\ (2.2)\qquad &= \alpha_2 - e^{\lambda_2(\hat{t}-1)} - \alpha_1 - 2(e^{\lambda_2(\hat{t}-1)} - e^{\lambda_2\hat{t}}) > (1/2)(\alpha_2-\alpha_1). \end{aligned}$$

Fix $m \leq K$ so that (2.2) is satisfied. For any $\delta > 0$ let $g_\delta(x)$ be a C^1-function such that $g_\delta(x) = h(x)$ for $x \notin (\hat{x}-\delta,\hat{x}+\delta)$ and

$0 \leq h(x) - g_\delta(x) < \delta$ for $x \in \mathbb{R}$. If $s' = s'(\delta)$ is such that $\alpha_2 - e^{\lambda_2 s'} = \hat{x}+\delta$, then $s'(\delta) \to \hat{t}$ as $\delta \to 0$. If $\phi_\delta(\theta) = \alpha_2 - e^{\lambda_2(s'(\delta)+\theta)}$, then $\phi_\delta \to \phi$ as $\delta \to 0$ in $C([-1,0],\mathbb{R})$. Let $y_\delta(t)$, $t \geq 0$, be the solution of (1.1(b,g_δ) through ϕ_δ and define $z_\delta(t) = \alpha_2 - e^{\lambda_2 t}$, $t < s'(\delta)$, $z_\delta(t) = y_\delta(t-s'(\delta))$, $t \geq s'(\delta)$. Then z_δ satisfies (1.1)(b,g_δ) on $(-\infty,\infty)$. By continuity, $z_\delta(t) \to z(t)$ as $\delta \to 0$, uniformly on compact intervals. Let $s'' = s''(\delta)$ be the smallest number such that $z_\delta(s''(\delta)) = \hat{x}-\delta$. Then $s''(\delta)$ exists and $s''(\delta) > s'(\delta)$. Moreover, since $z(\hat{t}+1) < \hat{x}$, it follows that $s''(\delta) < s'(\delta) + 1$ for $\delta > 0$ small enough. Hence, $s''(\delta) \to \hat{t}$ as $\delta \to 0$. For $\delta > 0$ small, it follows that $z_\delta(s''(\delta)+\theta) - \alpha_1$ is positive and nonincreasing. Moreover, $z_\delta(s''(\delta)-1) - \alpha_1 < \alpha - \alpha_2 < K$. Finally, by (2.2) and for δ small enough,

$$z_\delta(s''(\delta)-1) - \alpha_1 - 2(z_\delta(s''(\delta)-1) - z_\delta(s''(\delta)) \geq \rho.$$

Fixing such a small $\delta > 0$ and letting $f = g_\delta$ on $[\alpha_1,\alpha_2]$, $x(t) = z_\delta(t)$, we see from Lemma 2.2 that $\lim_{t\to\infty} x(t) = \alpha_1$, $\lim_{t\to\infty} x(t) = \alpha_2$. This proves the lemma.

Corollary 2.4. Let $a_1 < \cdots < a_{2k+1}$ be arbitrary real numbers, and, for every $i = 1,\ldots,k$, let m_{2i} be any negative number. Let $M > 0$ be arbitrary. Then there exists a $g \in G_k$, such that $g(a_i) = 0$ for $i = 1,\ldots,2k+1$, $g'(a_{2i}) = m_{2i}$ for $i = 1,\ldots,k$, g is affine on $(-\infty,a_1) \cup (a_{2k+1},\infty)$ and in a neighborhood of each a_i, $|g(x)| \leq M$ for $x \in [a_1,a_{2k+1}]$. Moreover, for every $i = 1,\ldots,k$ there are solutions $x_i(t)$ and $\bar{x}_i(t)$ of (1.1)(b,g) on $(-\infty,\infty)$ such that $a_{2i-1} < x_i(t) < a_{2i}$,

$a_{2i} < \bar{x}_i(t) < a_{2i+1}$, $t \in \mathbb{R}$, $x_i(+\infty) = a_{2i-1}$, $x_i(-\infty) = a_{2i}$, $\bar{x}_i(+\infty) = a_{2i+1}$, $\bar{x}_i(-\infty) = a_{2i}$.

We also need some lemmas on approximation of elements in G_k by analytic functions.

Lemma 2.5. For every $g \in G_k$ and every positive, continuous function $\phi\colon \mathbb{R} \to \mathbb{R}$ there exists an analytic function $h \in G_k$ such that

$$|g(x) - h(x)| + |g'(x) - h'(x)| < \phi(x), \quad x \in \mathbb{R}.$$

Proof: Let $a_1 < \cdots < a_{2k+1}$ be the zeros of g. There are numbers b_i^-, b_i^+, $i = 1,\ldots,2k+1$, $b_i^- < a_i < b_i^+$ such that $g(x) < 0$ for $x \le b_1^-$, $g(x) > 0$ for $x \ge b_{2k+1}^+$, $g'(x) \ne 0$ for $x \in [b_i^-,b_i^+]$, and the intervals $[b_i^-,b_i^+]$ are pairwise disjoint.

Let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a positive, continuous, function such that $\psi(x) < \phi(x)$ for $x \in \mathbb{R}$, ψ is integrable, $\psi(x) < \frac{1}{2}|g(x)|$ for $x \notin \bigcup_{i=1}^{2k+1} (b_i^-,b_i^+)$, $\psi(x) < \frac{1}{2}|g'(x)|$ for $x \in \bigcup_{i=1}^{2k+1} [b_i^-,b_i^+]$. By Whitney's Lemma ([3]), there is an analytic function $h\colon \mathbb{R} \to \mathbb{R}$ such that

$$|g(x) - h(x)| + |g'(x) - h'(x)| < \psi(x) \quad \text{for} \quad x \in \mathbb{R}.$$

We will show that $h \in G_k$. If $H(x) = \int_0^x h(s)ds$, $G(x) = \int_0^x g(s)ds$, then

$$H(x) = \int_0^x g(s)ds + \int_0^x (h(s) - g(s))ds$$

$$\geq G(x) - \int_0^x |h(s) - g(s)|ds$$

$$\geq G(x) - \int_0^x \psi(s)ds \geq G(x) - \int_{\mathbb{R}} \psi(s)ds \to \infty$$

as $|x| \to \infty$.

If $x \notin \bigcup_{i=1}^{2k+1} (b_i^-, b_i^+)$, then $|g(x)-h(x)| < \psi(x) < \frac{1}{2}|g(x)|$ and, hence, $h(x) \neq 0$. It follows that all zeros of h must be contained in $\bigcup_{i=1}^{2k+1} (b_i^-, b_i^+)$. If $x \in \bigcup_{i=1}^{2k+1} [b_i^-, b_i^+]$, then $|g'(x)-h'(x)| < \psi(x) < \frac{1}{2}|g'(x)|$ and, hence, $h'(x) \neq 0$. This means that h has at most one zero in $[b_i^-, b_i^+]$. Since $\text{sign } h(b_i^-) = \text{sign } g(b_i^-) \neq \text{sign } g(b_i^+) = \text{sign } h(b_i^+)$, it follows that h has exactly one zero in $[b_i^-, b_i^+]$ and this zero is simple. The Lemma is proved.

<u>Lemma 2.6</u>. <u>Let</u> $a_1 < \cdots < a_n$ <u>and</u> $\tilde{a}_1 < \cdots < \tilde{a}_n$ <u>be real numbers</u>. <u>Then there are</u> q, M, $0 < q < M$ <u>and there is an analytic mapping</u> $\alpha: \mathbb{R} \to \mathbb{R}$ <u>such that</u> α <u>is a diffeomorphism of</u> $\mathbb{R}$ <u>onto</u> $\mathbb{R}$, $q < \alpha'(x) < M$ <u>for</u> $x \in \mathbb{R}$, $\alpha(a_i) = \tilde{a}_i$, $i = 1,\ldots,n$.

<u>Proof</u>: Let $f: \mathbb{R} \to \mathbb{R}$ be a C^1-function such that $f'(x) > 0$ for $x \in \mathbb{R}$, f is affine on $(-\infty, a_1] \cup [a_n, \infty)$ and $f(a_i) = \tilde{a}_i$, $i = 1,\ldots,n$. Let $\gamma = \inf_{x \in \mathbb{R}} f'(x) > 0$ and $v_k(x) = \prod_{i \neq k} (x-a_i)$, $k = 1,\ldots,n$. By a simple calculation,

$$\tilde{M} := n \sup_k \{(\tanh(v_k(a_k)))^{-1}\} + 1$$

$$+ \sup_k \sup_{x \in \mathbb{R}} \left\{(\tanh(v_k(a_k)))^{-1} \cdot \frac{d(\tanh \circ v_k)(x)}{dx}\right\} < \infty .$$

By Whitney's Lemma, there is an analytic function $h: \mathbb{R} \to \mathbb{R}$ such that for $x \in \mathbb{R}$,

$$|h(x) - f(x)| + |h'(x) - f'(x)| < \gamma/3\tilde{M}.$$

If $\alpha = h + k$, where

$$k(x) = \sum_{k=1}^{n} \tanh(v_k(x)) \cdot \frac{f(a_k) - h(a_k)}{\tanh(v_k(a_k))}$$

then α is analytic,

$$|\alpha(x) - f(x)| \leq |h(x) - f(x)| + |k(x)| \leq \gamma/3\tilde{M} + \gamma/3.$$

Hence $\alpha(x) \to \pm\infty$ as $x \to \pm\infty$ and α is surjective. Also,

$$|\alpha'(x) - f'(x)| \leq |h'(x) - f'(x)| + |k'(x)| \leq \gamma/3\tilde{M} + \gamma/3$$

which implies $\sup_{x \in \mathbb{R}} |\alpha'(x)| < \infty$. Furthermore,

$$\frac{\gamma}{3} \leq f'(x) - \frac{\gamma}{3\tilde{M}} - \frac{\gamma}{3} \leq \alpha'(x).$$

Thus, $q < \alpha'(x) < M$, $x \in \mathbb{R}$ and α is a diffeomorphism. Finally, it is obvious that $\alpha(a_i) = \tilde{a}_i$, $i = 1,\ldots,n$, and the lemma is proved.

Corollary 2.7. *Let g and $\tilde{g}$ be two analytic functions belonging to G_k. Then there is a continuous mapping $H: [0,1] \to G_k$, such that $H(0) = \tilde{g}$, $H(1) = g$, and, for every $t \in [0,1]$, $H(t)$ is analytic.*

Proof: Let $a_1 < \cdots < a_n$ and $\tilde{a}_1 < \cdots < \tilde{a}_n$ be the zeros of g and $\tilde{g}$, resp. Let α be as in Lemma 2.6. Define for $t \in [0,1]$ and $x \in \mathbb{R}$

$$(H(t))(x) = \begin{cases} \tilde{g}((1-2t)x + 2t\alpha(x)) & , \text{ if } t \le 1/2 \\ (2-2t)\tilde{g}(\alpha(x)) + (2t-1)g(x), & \text{if } t > 1/2 \end{cases}$$

It is easy to show that H satisfies all requirements of the Corollary.

3. The main result.

In this section, we state and prove the main result.

Lemma 3.1. Let $g \in G_k$ and a be an unstable zero of g. If $x(t)$, $t \in (-\infty,\infty)$, is a solution of (1.1)(b,g) emanating from a, then either $x(t) > a$ for all t or $x(t) < a$ for all t.

Lemma 3.1 gives a limitation on the maximal number of equivalence classes that can occur in G_k. The remainder of the discussion is concerned with $k = 2$. There are at most five equivalence classes in G_2; namely, those described in the introduction. The main result is

Lemma 3.2. There is an analytic function in each of the equivalence classes $\{2[1,5],4[3,5]\}$, $\{2[1,3],4[1,5]\}$ in G_2.

Theorem 3.3. There are exactly five equivalence classes in G_2. Each equivalence class contains an analytic function.

Proof of Lemma 3.1:

Let $$V(\phi) = G(\phi(0)) + \frac{1}{2}\int_{-1}^{0} b'(\theta)\left[\int_{\theta}^{0} g(\phi(s))ds\right]^2 d\theta,$$

where $$G(x) = \int_0^x g(s)ds.$$

Then the derivative $\dot{V}(\phi)$ of V along solutions of (1.1) (b,g) is given by

$$(3.1)\qquad \dot{V}(\phi) = -\frac{1}{2}b'(-1)\left[\int_{-1}^{0} g(\phi(\theta))d\theta\right]^2 - \frac{1}{2}\int_{-1}^{0} b''(\theta)\left[\int_{\theta}^{0} g(\phi(s))ds\right]^2 d\theta,$$

(see [1]).

Since $\lim_{t \to -\infty} x(t) = a$, it follows that $\lim_{t \to -\infty} V(x_t) = G(a)$.

If the Lemma is not true, there exists a $t_0 > 0$ such that $x(t_0) = a$. By (3.1), $V(x_{t_0}) < G(a)$. Hence $\frac{1}{2}\int_{-1}^{0} b'(\theta)[\int_{\theta}^{0} g(\phi(s))ds]^2 d\theta < 0$, a contradiction which proves the Lemma.

Proof of Lemma 3.2: We will show that there is a $g \in G_2$ representing class $\{2[1,5],4[3,5]\}$. Symmetry in the proof to follow implies that there is a $g' \in G_2$ representing class $\{2[1,3],4[1,5]\}$. Since these classes are stable classes, Lemma 2.5 implies that g and g' can be chosen analytic, which proves the lemma. Choose $a_1 < a_2$ arbitrarily. Define $f \in C^1$ such that f has exactly two zeros, a_1 and a_2, f is affine in a neighborhood of a_1, a_2 and on $(-\infty,a_1) \cup (a_2,\infty)$, $f'(a_1) > 0$, $f'(a_2) = m_2 < 0$. There is a unique $\lambda_2 > 0$ such that $x(t) = a_2 + e^{\lambda_2 t}$, $t \in (-\infty,\infty)$ is a solution of (1.1)(b,f). Let $t_0 > 0$ be arbitrary and let $y(t)$, $t \in [0,1]$, be defined as

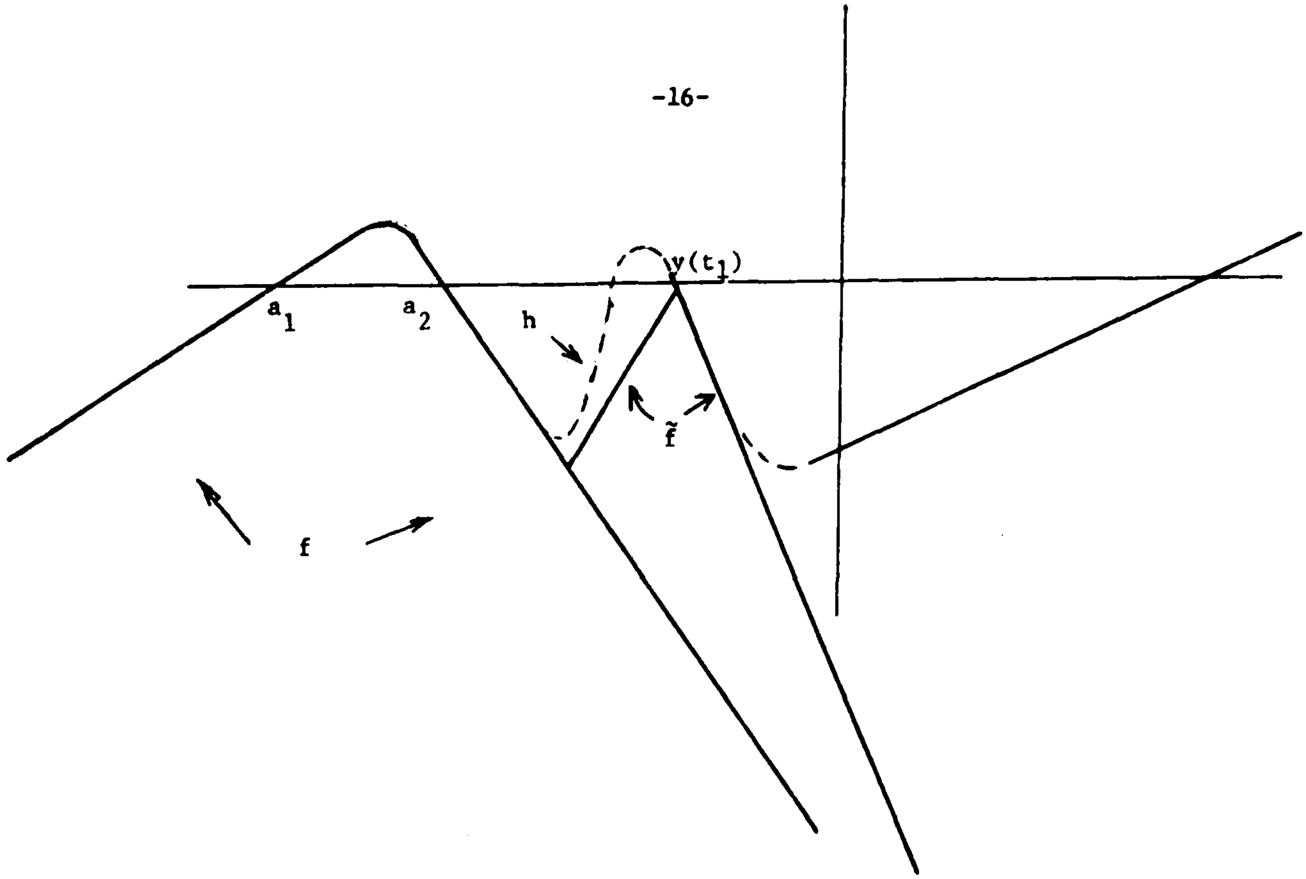

Figure 2

$$y(t) = x(t_0) - \int_0^t \left(\int_{-1}^{-s} b(\theta) f(x(s+\theta))\, d\theta \right) ds$$

Hence, there is a $0 < t_1 < 1$ such that $\dot{y}(t) > 0$ for $t \in [0,t_1]$ and, so, $y(t_1) > x(t_0)$. Define $\tilde{f}$ such that $\tilde{f} = f$ on $(-\infty, x(t_0)]$, $\tilde{f}$ is affine on $[x(t_0), y(t_1)]$, $\tilde{f}(y(t_1)) = 0$, and $\tilde{f}$ is affine on $[y(t_1), \infty)$ with negative slope. If $\phi(\theta) = x(t_0+\theta)$, $\theta \in [-1,0]$, let $\tilde{y}$ be the solution of (1.1)$(b,\tilde{f})$ through ϕ, $t \geq 0$. Then obviously $\tilde{y}(t) > y(t)$ for $t \in [0,t_1]$. If we define $\tilde{x}(t) = x(t)$, $t \leq t_0$, $\tilde{x}(t) = \tilde{y}(t-t_0)$, $t > t_0$, then $\tilde{x}$ satisfies (1.1)$(b,\tilde{f})$ on $(-\infty,\infty)$. Moreover, there is an $s_1 > s_0$ such that $\tilde{x}(s_1+\theta) > y(t_1)$ for $\theta \in [-1,0]$, and $\tilde{x}(s_1+\theta)$ is nondecreasing on $[-1,0]$. Now we perturb $\tilde{f}$ a little to obtain a C^1-map

h which coincides with $\tilde{f}$ on $[y(t_1),\infty)$, and h has four simple zeros, $a_1 < a_2 < a_3 < a_4$, where $a_3 \in (x(t_0),y(t_1))$, $a_4 = y(t_1)$ and there is a solution $z(t)$ of (1.1)(b,h) on $(-\infty,\infty)$ such that $z(t) \to a_2$ as $t \to -\infty$ and, for some $s_2 \in \mathbb{R}$, $z(s_2+\theta) > y(t_1)$, $\theta \in [-1,0]$, $z(s_2+\theta)$ is non-decreasing on $[-1,0]$ (cf. Fig. 2). Now the application of Lemma 2.2 and the argument from the proof of Lemma 2.3 easily completes the proof of the theorem.

<u>Proof of Theorem 3.3</u>: Let g be an analytic function representing class $\{2[1,5],4[3,5]\}$. By Corollary 2.4, there exists an $f \in G_2$ representing class $\{2[1,3],4[3,5]\}$. Since the class $\{2[1,3],4[3,5]\}$ is ~-stable, we infer from Lemma 2.5 that there is an analytic function $\tilde{g} \in G_2$ representing $\{2[1,3],4[3,5]\}$. By Corollary 2.7, there is a continuous map $H: [0,1] \to G_2$ such that $H(0) = \tilde{g}$, $H(1) = g$ and $H(s)$ is an analytic function for every fixed s. Let $a_1(s) < a_2(s) < a_3(s) < a_4(s) < a_5(s)$ be the zeros of $H(t)$. It follows that $a_i(s)$ is continuous for $i = 1,\ldots,5$.

Let $s_0 = \sup\{s \in [0,1]$: the solution $x(s',t)$, $t \in (-\infty,\infty)$ of (1.1)(b,H(s')) emanating from $a_2(s')$ and staying to the right of $a_2(s')$ is such that $\lim_{t\to\infty} x(s',t) = a_3(s')$, for every $0 \leq s' \leq s\}$. It follows easily that $0 < s_0 < 1$ and that $\lim_{t\to\infty} x(s_0,t) \notin \{a_3(s_0),a_5(s_0)\}$. Hence $\lim_{t\to\infty} x(s_0,t) = a_4(s_0)$, i.e., $H(s_0)$ represents the saddle connection $\{2[1,4],4[3,5]\}$. By using g' instead of g where g' is analytic and represents class $\{2[1,3],4[1,5]\}$, we analogously prove that there is an analytic function $\in G_2$ representing the saddle connection $\{2[1,3],4[2,5]\}$. This completes the proof of Theorem 3.3.

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